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An equilibrium penny-shaped crack in an inhomogeneous elastic medium☆

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ABSTRACT

The problem of a penny-shaped tensile crack in a continuously-inhomogeneous space is considered. The problem reduces to a dual integral equation for which an approximate analytical solution is constructed. It is proved that the approximate solution of the integral equation is asymptotically exact for both small and large values of the dimensionless geometric parameter of the problem. The accuracy of the solution obtained is investigated. Expressions are presented for the stress intensity factor, the energy of the opening of the crack, the displacements of its sides and the normal components of the stress tensor in the neighbourhood of its contour. In the numerical analysis of the solution of the problem, special attention is paid to analysing of the problem when the first derivative of the change in the elastic properties of the material changes sign.

A large number of papers have been published in which problems in crack theory are considered for inhomogeneous materials. Cases when the gradient of the change in the properties of the material does not change sign have been investigated and, in most of the papers, inhomogeneity laws are considered when the variable shear modulus or Poisson's ratio vanishes or becomes infinite at at least one point of the body.

The majority of papers are concerned with plane or antiplane crack problems. For instance, crack problems in an inhomogeneous plane have been considered for linear,[1] power,[2] hyperbolic,[2,3] and exponential,[4–6] Lamé coefficient relations as well as for certain other cases.[7,8]

A lesser number of papers have dealt with penny-shaped crack problems. The classical problem for a penny-shaped crack in a homogeneous, isotropic, elastic medium was investigated for the first time by Sneddon[9] using the method of dual integral equations and by Sack[10] who used spherical harmonic functions. The problem has been considered for two-layer materials.[11–15] It has been treated[16,17] under the assumption that Poisson's ratio of an elastic material is constant and the shear modulus (or Young's modulus) varies as a power or exponential law. The situation when Young's modulus or the shear modulus in a material vanishes or becomes infinite is physically unreal. In reality, Young's modulus (or the shear modulus) may vary from point to point, changing from one finite value to another, and is non-zero at infinity.

Crack problems in a continuously inhomogeneous elastic medium have been considered.[6,18] A Griffith crack problem in an isotropic elastic medium with a shear modulus that varies with the distance to the plane of the crack has been investigated.[18] A class of problems for a plane crack on the interface of inhomogeneous elastic media under the action of an internal pressure has been investigated.[19] The effect of a change in the properties of a material on the specific features of the development of a transverse shear crack as a function of the spatial variables was investigated numerically in Ref. 20. The eigenvalue scheme, which is used to investigate homogeneous materials, has been modified with the aim of taking account of the change in the properties of a material in a direction which is symmetrical about the plane of the crack.

Layers with a continuous variation in properties have been considered.[21] An estimate of the bonds with such layers and the optimization of the condition for gluing them together was based by the authors on an analysis of the stresses and, in the case of the development of a crack, on an analysis of the stress intensity factor (SIF). The SIFs in continuously-inhomogeneous materials were calculated for several special cases.

Whereas, in homogeneous materials the SIF is a function of the size and curvature of the contour of the plane crack, the applied mechanical load and/or thermal action, in continuously inhomogeneous materials the SIF additionally depends on the change in Young's modulus of the component parts of the material. The first results for the solution of the crack problem in an inhomogeneous material using

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a weighting function were obtained by Fett and Munz.[22] An axisymmetric problem was considered[23] for a penny-shaped crack located in a plane which is the boundary for two similar elastic inhomogeneous half-spaces with a continuous elastic surface inhomogeneity, the shear modulus of each of which varies as $G(z) = G_1 + G_2 e^{\pm z\varsigma}$. The solution of the equilibrium equations for this type of inhomogeneity was found using Hankel transforms. The problem was reduced to a Fredholm equation of the second kind which was solved numerically. The effect of the amplitude of the opening of a crack on the SIF in the neighbourhood of its contour was investigated.

The present investigation is close in its formulation of the problem to those in Refs 24–26 in which plane problems in crack theory were considered for piecewise-homogeneous materials. Unlike in these papers, a penny-shaped crack problem in a continuously inhomogeneous material is considered below in the case of an arbitrary variation of the elastic properties (special attention is given to the case when the gradient of the change in the properties of the material changes sign). A method, proposed earlier,[27] is developed to solve it.

A special case of a model of an inhomogeneous medium, investigated in this paper, has been treated[28,29] under the assumption that the shear modulus of the medium is constant and Poisson's ratio varies in an arbitrary way with distance from the plane of the crack. The investigation was based on the fact that, in this case, the transform of the kernel of the dual integral equation of the crack problem can be successfully constructed in an explicit form. Note that, by virtue of the physical constraints, Poisson's ratio can only vary over a range from −1 to 1/2. This means that such a model of a medium is weakly innhomogeneous. The solution of the dual integral equation was constructed numerically[28,29] and, for this type of integral equations, the numerical solution is effective over a narrow range of values of the characteristic geometric parameter close to unity, for which the examples are considered.

1. Formulation of the boundary value problem and some properties of the transform of the kernel of the dual integral equation

An axially static problem is considered for a penny-shaped crack of normal separation in an elastic inhomogeneous isotropic space. The cylindrical system of coordinates (r, φ, z) is associated with the space. A crack of radius R is located in the $z = 0$ plane and its centre coincides with the origin of the coordinates.

Young's modulus in the space is an arbitrary continuous (or piecewise-continuous) symmetric function of the distance from the plane of the crack $(z = 0)$:

$$E(z) = E_0 f(|z|), \quad 0 \le |z| \le H; \quad E(z) = E_0 f(|H|), \quad |z| > H \tag{1.1}$$

and the conditions

$$\min_{z\in(0;\infty)} \Delta(z) \ge c > 0, \quad \max_{z\in(0;\infty)} \Delta(z) \le c < \infty, \quad \lim_{z\to\infty} \Delta(z) = \text{const}$$
$$\Delta(z) = 2\mathrm{M}(z)(\Lambda(z) + \mathrm{M}(z))(\Lambda(z) + 2\mathrm{M}(z))^{-1} = G(z)(1 - \nu(z))^{-1} = E(z)(1 - \nu^2(z))^{-1}/2 \tag{1.2}$$

are satisfied. Here, $G(z)$ is the shear modulus, $\Lambda(z)$ and $\mathrm{M}(z)$ are Lamé coefficients and $\nu(z)$ is Poisson's ratio of the inhomogeneous space. The sides of the crack are loaded from within by a distributed pressure $p(r)$. When account is taken of the symmetry of the elastic properties of the material and the loading conditions with respect to the plane in which the crack is located, we arrive at the following mixed boundary value problem for the half-space $z \ge 0$. The condition

$$z = 0: \quad \tau_{rz} = 0, \quad 0 < r < \infty; \quad \sigma_z = -p(r), \quad 0 \le r \le R; \quad w(r) = 0, \quad r > R \tag{1.3}$$

holds on the boundary $z = 0$. We will assume that, when $|z| = H$, the conditions for matching the strains and stresses are satisfied and that the stresses and strains tend to zero at infinity.

It has been established[30] that, when conditions (1.2) are satisfied, the following relation between the normal displacements σ_z and the normal strains $w(r)$ of the surface of the half-space $(z = 0)$ holds in the case of a continuously inhomogeneous medium

$$w(r,0) = \frac{1}{\Delta(0)} \int_0^R q(\rho)\rho d\rho \int_0^\infty L(\alpha) J_0(\alpha\rho) J_0(\alpha r) d\alpha \tag{1.4}$$

Here,

$$q(r) = \sigma_z(r,0) = \int_0^\infty Q(\alpha) J_0(\alpha r)\alpha d\alpha, \quad Q(\alpha) = \int_0^R q(\rho) J_0(\alpha\rho)\rho d\rho$$

The function $L(\alpha)$ is found numerically by the method of modulating functions[30] and possesses the following properties[30,31] when conditions (1.1) and (1.2) are satisfied

$$L(\alpha) = A + B|\alpha| + o(\alpha^2), \quad \alpha \to 0 \tag{1.5}$$

$$L(\alpha) = 1 + D|\alpha|^{-1} + o(\alpha^{-2}), \quad \alpha \to \infty \tag{1.6}$$

Here, $A = \lim_{|z|\to\infty} \Delta(0)/\Delta(z)$, B, D are certain constants. Note that the relations

$$L(\alpha) = A + o(\alpha), \quad A = D_1^{-1} D_2^{-1} \cdots D_{n-1}^{-1}, \quad \alpha \to 0; \quad D_k = \frac{E_{k+1}(1 - \nu_k^2)}{E_k(1 - \nu_{k+1}^2)}; \quad n \ge 2 \tag{1.7}$$

$$L(\alpha) = 1 + (\alpha^2 h_1^2 + \alpha h_1) M e^{-2\alpha h_1} + o(e^{-2\alpha h_1}), \quad \alpha \to \infty$$

$$M = \frac{4(\tilde{\Delta}_1 + k_2)^2 - 1}{(D_1+1)^2 - (k_1 D_1 - k_2)^2}, \quad \tilde{\Delta}_1 = \frac{D_1}{2(1-\nu_1)}, \quad k_j = \frac{1-2\nu_j}{2(1-\nu_j)}, \quad j = 1,2,\ldots \tag{1.8}$$

hold in the case of a multilayer packet. Here h_1 is the thickness of the upper elastic layer, and E_i and ν_i are the values of Young's modulus and Poisson's ratio in the i-th layer.

Expressions (1.6) and (1.8) differ in the nature of the behaviour of the second term of the expansion when $\alpha \to \infty$. The difference in the properties of the solutions of the integral equations for continuously inhomogeneous and multilayer media result from this.[31] Properties (1.5) and (1.7) indicate that the value of $L(0)$ is independent of how the Lamé coefficients change in the interval from $z=-H$ to $z=H$ and is solely determined by their values at $z=0$ and $|z|=H$. This has been noted earlier in the case of a multilayer medium.[32] We will use an approach described earlier (Ref. 33, pp. 58–59). Using the methods of operational calculus and taking account of condition (1.3), relation (1.4) can be rewritten in the form

$$\int_0^R \delta(\rho)\rho d\rho \int_0^\infty \frac{\alpha^2}{L(\alpha)} J_0(\alpha\rho) J_0(\alpha r) d\alpha = -\frac{p(r)}{\Delta(0)}, \quad 0 \le r \le R; \quad \delta(r) = -w(r,0) \tag{1.9}$$

The function $\delta(r)$ describes the shape of the crack in the plane. The natural conditions for the closing of the sides of the crack

$$\delta(R) = 0 \tag{1.10}$$

hold. We assume that the function $\delta(r)$ can be extended in an even manner and that

$$\delta(r) = \delta(-r), \quad \delta''(0) = 0 \tag{1.11}$$

2. Some auxiliary transforms

Integrating by parts and using condition (1.10), Eq. (1.9) can be written in a somewhat different form and it can be concluded that, if the displacements of the sides of the crack are known, the normal stresses in the plane of the crack on its continuation are given by the expression

$$p(r) = \Delta(0) \int_0^R \delta'(\rho) d\rho \int_0^\infty \frac{\alpha\rho}{L(\alpha)} J_1(\alpha\rho) J_0(\alpha r) d\alpha, \quad r > R \tag{2.1}$$

We act on the right-hand and left-hand sides of equality (2.1) with the operator

$$B[\psi] \equiv \int_0^r \rho\psi(\rho) d\rho$$

and change to dimensionless variables, introducing the notation

$$\alpha H = u, \quad \lambda = \frac{H}{R}, \quad r' = \frac{r}{R}, \quad \rho' = \frac{\rho}{R}, \quad \delta'(\rho' R) = \varphi'(\rho'), \quad \frac{u}{\lambda} = \beta$$

$$q^*(r) = \frac{1}{Rr} \int_0^r p(\rho)\rho d\rho = \frac{1}{Rr'} \int_0^{r'} p(R\rho')'\rho R d'\rho, \quad 0 \le r' \le 1$$

We will hence forth omit the prime. Finally, the problem reduced to solving of the integral equation

$$\int_0^1 \varphi(\rho) d\rho \int_0^\infty \frac{\rho J_1(\beta\rho) J_1(\beta r)}{L(\beta\lambda)} d\beta = \frac{1}{\Delta(0)} q^*(r), \quad 0 \le r \le 1 \tag{2.2}$$

We now introduce the notation

$$\Delta_1(\beta) = \int_0^1 \varphi'(\rho) J_1(\beta\rho)\rho d\rho, \quad p^*(r) = \frac{1}{r}\int_0^r p(\rho)\rho d\rho \tag{2.3}$$

Taking account of the fact that $\varphi(r)=0$ when $r>1$ and, consequently, $\varphi'(r)=0$ when $r>1$, we have, on the basis of the properties of the Hankel transforms,

$$\int_0^\infty \Delta_1(\beta)\beta J_1(\beta\rho) d\rho = \begin{cases} \varphi'(\rho), & 0 < \rho \le 1 \\ 0, & \rho > 1 \end{cases} \tag{2.4}$$

Using relations (2.2)–(2.4), we obtain the dual integral equation (DIE)

$$\int_0^\infty \frac{\Delta_1(\beta)}{L(\beta\lambda)} J_1(\beta r)d\beta = \frac{p^*(r)}{\Delta(0)}, \quad 0 \le r \le 1$$
$$\int_0^\infty \Delta_1(\beta)\beta J_1(\beta r)d\beta = 0, \quad r > 1 \tag{2.5}$$

If the function $\Delta_1(\beta)$ is found, then $\delta(r)$ can be determined using the following relations

$$\delta'(r) = \int_0^\infty \Delta_1(\alpha)\alpha J_1(\alpha r)d\alpha$$
$$\delta(r) = \int_R^r \delta'(\rho)d\rho = \int_0^\infty \Delta_1(\alpha)\alpha\int_R^r J_1(\alpha\rho)d\alpha d\beta = \int_0^\infty \Delta_1(\alpha)J_0(\alpha r)d\alpha \tag{2.6}$$

(condition (1.10) has been used).

Since the problem considered is axisymmetric, the condition

$$p(r) = p(-r)$$

holds. Consequently, the function $p^*(r)$ is odd and can be expanded in a Dini–Bessel series (Ref. 34, p. 206)

$$p^*(r) = \sum_{i=1}^\infty C_i J_1(\mu_i r)$$

$$\sum_{i=1}^\infty |C_i\lambda_i| \le M_1(-1,1) < \infty, \quad C_i = 2\left(1 + \frac{\alpha^2-\beta^2}{\beta^2\mu_i^2}\right)^{-1} J_1^2(\mu_i)\int_0^1 rp^*(r)J_1(\mu_i r)dr \tag{2.7}$$

Here, $\mu_1, \mu_2, \ldots$ are the positive roots of the equation

$$\alpha J_1(r) + \beta r J_1'(r) = 0,$$

arranged in increasing order, $\alpha/\beta+1 > 0$, α and β are given real numbers and $M_1(-1,1)$ is a certain constant.

3. Description of the bilaterally asymptotic method of solving the dual integral equation

Definition 1. The function $F(\gamma)$ belongs to the class Π_N (to the class Σ_M or $S_{N,M}$) if it has the form

$$F(\lambda\gamma) = \prod_{i=1}^N (\gamma^2 + b_i^2\lambda^{-2})(\gamma^2 + a_i^2\lambda^{-2})^{-1} \equiv F_N(\gamma\lambda) \in \Pi_N \tag{3.1}$$

$$F(\lambda\gamma) = \sum_{k=1}^M C_k\lambda^{-1}|\gamma|(\gamma^2 + D_k^2\lambda^{-2})^{-1} \equiv F_M^\Sigma(\gamma\lambda) \in \Sigma_M \tag{3.2}$$

respectively. Here, a_i,

$$F(\lambda\gamma) = F_N(\gamma\lambda) + F_M^\Sigma(\gamma\lambda) \in S_{N,M} \tag{3.3}$$

respectively. Here, a_i, b_i ($i = 1,2,\ldots,N$), c_k, d_k ($k = 1,2,\ldots,M$) are certain constants such that $(a_i - a_k)(b_i - b_k) \ne 1$ when $i \ne k$.

We now consider the transform of the kernel $L^{-1}(\beta\lambda) = F(\lambda\alpha)$ of the DIE (2.5). The function $F(\lambda\alpha)$ possesses the properties (1.5) and (1.6), and $A = \Delta(|H|)/\Delta(0)$. It is well known[35] that, if a function possesses the properties (1.5) and (1.6), it can be approximated by expressions of the form

$$F(\alpha\lambda) = F_N(\alpha\lambda) + F_\infty^\Sigma(\alpha\lambda) \tag{3.4}$$

Below, we shall also denote the integral operator, which corresponds to the function $F(\lambda\alpha)$ and belongs to the class X, by X.

Using approximation (3.4), we rewrite DIE (2.5) in the operator form

$$\Pi_N\delta + \Sigma_\infty\delta = p^* \tag{3.5}$$

The operator Π_N corresponds to DIE (2.5) of the function $F(\lambda\alpha)$ of the form (3.1) and Σ_∞ to the function $F(\lambda\alpha)$ of the form of (3.2).

Theorem 1. (*Ref. 35*). *If $p^*(x)$ is an odd function such that representation (2.7) holds, the operator Π_N is invertible and the estimate*

$$\|\delta_N\|_{C^{(0)-}_{1/2}(-1,1)} \le m(\Pi_N)M_p(-1,1), \quad m(\Pi_N) = \text{const} \tag{3.6}$$

holds.

In the general case, the following assertion is applicable to Eq. (3.5).

Theorem 2. *(Refs 23 and 24). When conditions (1.2) are satisfied, Eq. (2.5) is uniquely solvable in the space $C^{(0)-}_{1/2}(-1,1)$ for the functions $p^*(x)$ which have the representation (2.7) when $0<\lambda<\lambda^*$ and $\lambda>\lambda^0$, where λ^* and λ^0 are certain fixed values of λ. At the same time, the estimate*

$$\|\delta^*(x)\|_{C^{(0)-}_{1/2}(-1,1)} \le m(\Pi_N,\Sigma_\infty)M_{p^*}(-1,1) \tag{3.7}$$

holds.

It follows from the results obtained earlier[36,37] that, when $\lambda\to 0$ and $\lambda\to\infty$, the operator $\prod_N^{-1}\Sigma_\infty$ of Eq. (3.5) is a contraction operator.[38] In other words, the analytical solution of this equation will be bilaterally asymptotically exact when $\lambda\to 0$ and $\lambda\to\infty$. In this case, the error in the approximate solution does not exceed the error in approximating the functions $F(\lambda\alpha)$ by functions of the class Π_N. Note that the fact that there are no analytical solutions for problems in the theory of cracks in continuously inhomogeneous media has been pointed out in Ref. 39.

4. Closed solution of an auxiliary dual integral equation

Consider the operators

$$V_1^t[\varphi(r)] = \frac{d}{dt}\left(t\int_0^t \frac{\varphi(r)dr}{\sqrt{t^2-r^2}}\right), \quad V_1^t[J_1(r\beta)] = \sin(t\beta) \tag{4.1}$$

$$V_2^t[\varphi(r)] = \frac{d}{dt}\left(t\int_t^\infty \frac{\varphi(r)dr}{\sqrt{t^2-r^2}}\right), \quad V_2^t[J_1(r\beta)] = \frac{\sin(t\beta)}{t\beta} \tag{4.2}$$

Acting on the first equation of (2.5) with the operator (4.1) and on the second equation with the operator (4.2), we obtain the DIE:

$$\begin{aligned}
&\int_0^\infty \Delta_1(\alpha)F(\alpha\lambda)\sin(t\alpha)d\alpha = g(t), \quad 0\le t\le 1; \quad g(t) = \frac{1}{\Delta(0)}\frac{d}{dt}\left(t\int_0^t \frac{p^*(r)dr}{\sqrt{t^2-r^2}}\right)\\
&\int_0^\infty \Delta_1(\alpha)\alpha\sin(t\alpha)d\alpha = 0, \quad t>1
\end{aligned} \tag{4.3}$$

Suppose the function $F(\lambda\alpha)$ has the form (3.1)

$$F(\lambda\alpha) = \prod_{i=1}^N (\gamma^2 + b_i^2\lambda^{-2})(\gamma^2 + a_i^2\lambda^{-2})^{-1} = L_N^{-1}(\lambda\alpha) \tag{4.4}$$

Using the methods of the operational calculus in a similar way to the approach described earlier,[35] it is easy to obtain the general solution of DIE (4.3).

Finally, for the displacements of the sides of the cracks we obtain

$$\begin{aligned}
&\delta(r) = \frac{2}{\pi}\left(-\frac{p}{\Delta(0)}\right)\left[L_N(0)\sqrt{1-r^2} + \Sigma_\infty(r) + \sum_{i=1}^N C_i\tilde{b}_i\int_r^1 \frac{\operatorname{sh}\tilde{b}_i t dt}{\sqrt{t^2-r^2}}\right]\\
&\Sigma_\infty(r) = L_N(\lambda\mu_k)C_k\int_r^1 \frac{\sin\mu_k t dt}{\sqrt{t^2-r^2}}
\end{aligned} \tag{4.5}$$

The constants C_i are determined from the following system of linear algebraic equations

$$\begin{aligned}
&\sum_{i=1}^N C_i \mathrm{P}\left(\frac{a_k}{\lambda};\frac{b_i}{\lambda}\right) + \frac{1+a_k\lambda^{-1}}{a_k^2\lambda^{-2}}L_N(0) + \sum_{j=1}^\infty C_j L_N(\lambda\mu_j)\mathrm{D}\left(\frac{a_k}{\lambda};\mu_j\right) = 0, \quad k = 1,2,\ldots,N\\
&\mathrm{P}(A;B) = \frac{B\operatorname{ch}B + A\operatorname{sh}B}{A^2-B^2}, \quad \mathrm{D}(A;\mu) = \frac{\mu\cos\mu + A\sin\mu}{A^2+\mu^2\lambda^2}
\end{aligned} \tag{4.6}$$

In the special case when a uniform pressure is applied within the crack, the displacement of the sides of the crack is determined using formula (4.5) when $\Sigma_\infty(r)=0$.

We now transform expression (2.1) representing the normal stresses in the plane of the crack. Using the first relation of (2.6), we have

$$p(r) = -\frac{p}{\Delta(0)}\int_0^\infty \alpha\Delta_1(\alpha)F_N(\lambda\alpha)J_0(\alpha r)d\alpha, \quad r > 1$$

and, after elementary reduction, we obtain

$$p(r) = \frac{2}{\pi}p\left\{\arcsin\frac{1}{r} - \left[\frac{L(0)}{\Delta(0)} + \sum_{n=1}^{N} C_n \operatorname{sh}\tilde{b}_n\right]\frac{1}{\sqrt{r^2-1}} - \right.$$
$$\left. - \sum_{k=1}^{N} L_N^k(\tilde{a}_k)\left[\frac{L(0)}{\Delta(0)} + \sum_{n=1}^{N} C_n \frac{\tilde{a}_k^2 \operatorname{sh}\tilde{b}_n}{\tilde{a}_k^2 - \tilde{b}_k^2}\right]\int_1^r \frac{\exp(-\tilde{a}_k(t-1))}{\sqrt{r^2-t^2}}dt\right\}, \quad r > 1 \tag{4.7}$$

Using expression (4.7), we find the normal stress intensity factor

$$K_I = \lim_{r\to 1+0}\sqrt{r-1}\,p(r) = -\frac{\sqrt{2}}{\pi}\frac{p}{\Delta(0)}\left(L(0) + \Delta(0)\sum_{i=1}^{N} C_i \operatorname{sh}\tilde{b}_i\right) \tag{4.8}$$

We now determine the energy of opening the crack in the case of a uniform internal pressure[39]

$$A = p\int_0^1 2\pi r\delta(r)dr = 4p^2\left[\frac{1}{3\Delta(0)}L_N(0) + \sum_{i=1}^{N} C_i \frac{\tilde{b}_i \operatorname{ch}\tilde{b}_i - \operatorname{sh}\tilde{b}_i}{\tilde{b}_i^2}\right] \tag{4.9}$$

(expression (4.5) when $\Sigma_\infty(r) = 0$ has been used).

5. Numerical results

We will now consider a penny-shaped crack in a space in the case when the pressure within the crack is uniformly distributed and is of unit intensity. Poisson's ratio ν=1/3, $0 \le |z| < \infty$. We will consider the following laws of inhomogeneity (see formula (1.1) and Fig. 1)

$$f_1(\bar{z}) = \frac{7}{2}, \quad f_2(\bar{z}) = \frac{2}{7}, \quad f_3(\bar{z}) = \frac{7}{2} + \frac{5}{2}\bar{z}, \quad f_4(\bar{z}) = \frac{2}{7} - \frac{5}{7}\bar{z}$$
$$f_5(\bar{z}) = 1 + \frac{5}{2}\sin^2(\bar{z}\pi), \quad f_6(\bar{z}) = \frac{1}{f_5(\bar{z})}; \quad \bar{z} = \frac{z}{H} \tag{5.1}$$

The ratio of the transformants $F_n(\alpha)$ to the transformant $F_0(\alpha)$ is shown in Fig. 2 for a homogeneous material (the number of a curve corresponds to the number of the law of inhomogeneity (5.1)). As a result of direct calculations, it was established that the relative error in approximating the kernel transformant function $F_N(\alpha)$, constructed numerically, and the function $F_n(\alpha)$ calculated using formula (3.1)

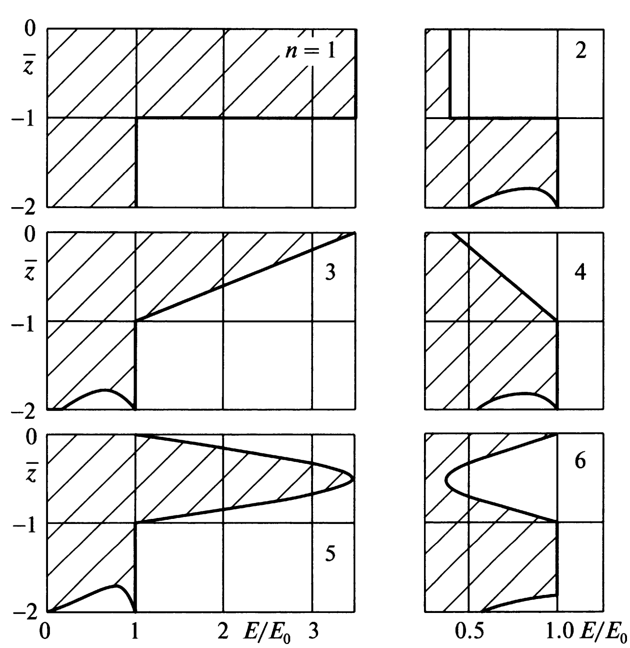

Fig. 1.

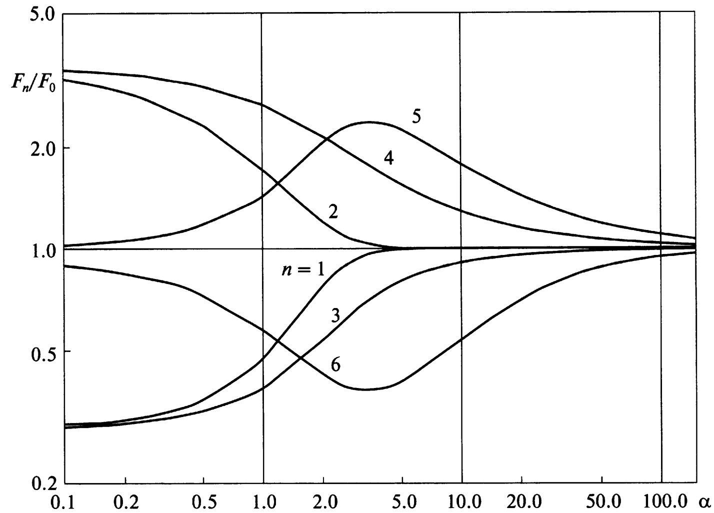

Fig. 2.

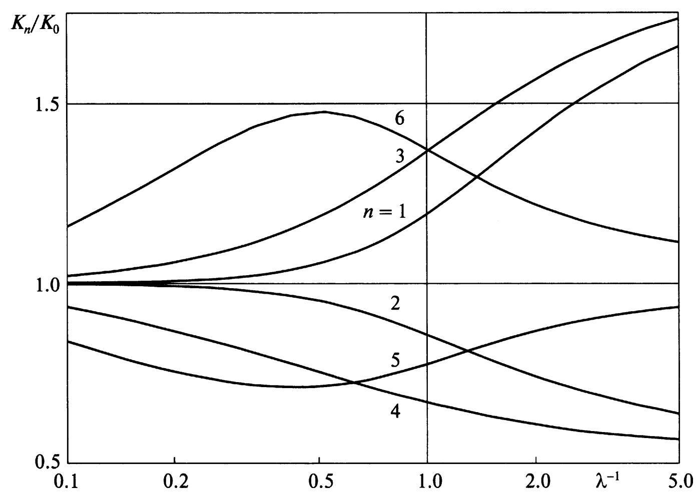

Fig. 3.

does not exceed 2–3% in the case of monotonic laws of innhomogeneity. In the case when the first derivative of the function describing the elastic properties of the medium changes sign an in inhomogeneous layer, the error of approximation does not exceed 5.5%. Consequently, on the basis of Theorem 2, the error in the approximate results obtained does not exceed the error in approximating the functions $F_n(\alpha)$ by functions of the class Π_N.

The relative values of the normal stress intensity factors K_n/K_0 are shown in Fig. 3 as a function of the reduced size of the crack λ^{-1} for different of forms of inhomogeneity. The values of the normal stress intensity factors $K_n(\lambda^{-1})$ were calculated using formula (4.8) for the different inhomogeneity laws (5.1). K_0 is the normal stress intensity factor in a homogeneous space. The number of a curve corresponds to the number of the inhomogeneity law (5.1).

The change in the normal stress intensity factors K_m/K_0 is shown in Fig. 4 as a function of λ^{-1} for different values of m in a representation for the inhomogeneities $f_5(z)$ and $f_6(z)$ of the form

$$\tilde{f}_5(\bar{z}) = 1 + (m-1)\sin^2(\bar{z}\pi), \quad \tilde{f}_6(\bar{z}) = \frac{1}{\tilde{f}_5(\bar{z})} \tag{5.2}$$

where m is the ratio of Young's modulus on the free surface of the crack E_0 and the maximum (minimum) Young's modulus inside the inhomogeneous layer $E(\pm 0.5)$ (see Fig. 1). These representations simulate the case of the continuous alternation of layers of different stiffness within the inhomogeneous material. In particular, an elementary smooth function $f_5(z)$ was chosen for Young's modulus, the derivative of which changes sign in the neighbourhood of the crack and the function $f_6(z)$ which is symmetric to it with respect to $E_0 = 1$. In Fig. 4, the three upper curves correspond an inhomogeneity of the form of $f_6(z)$ and the three lower curves to $f_5(z)$. The number of a curve corresponds to the value of m from (5.2). It should be noted that a value of $m = 3.5$ corresponds to the fifth and sixth curves ($n = 5$, $n = 6$) in Fig. 3.

The dependence of the relative energy of opening the crack on λ^{-1} is shown in Fig. 5 for inhomogeneity laws (5.1). It is clear that, if the first derivative of the inhomogeneity law changes sign, crack sizes exist for which their development takes place without the influx of additional energy.

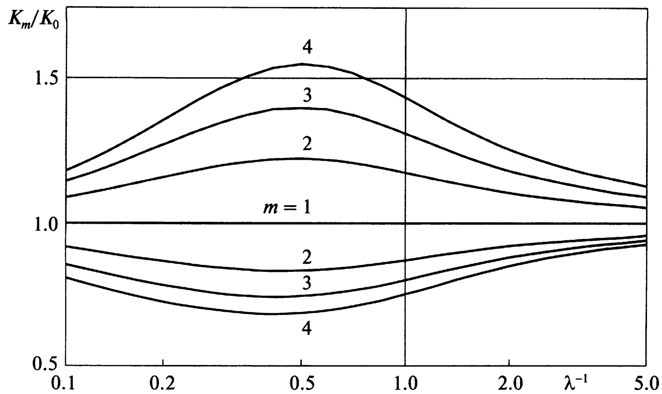

Fig. 4.

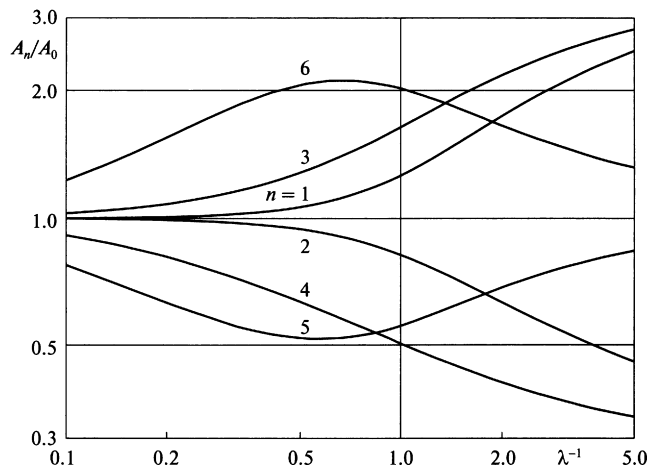

Fig. 5.

The fact that the solution in the form of (4.9), as well as the quantities represented by the corresponding curves, are of an asymptotically exact character both when $\lambda \to 0$ and when $\lambda \to \infty$ is illustrated numerically in Figs. 3–5.

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References

1. Rogers C, Clements DL. Bergman's intrgral operator method in inhomogeneous elasticity. *Quat Appl Math* 1978;**36**(3):315–21.
2. Clements DL, Atkinson C, Rogers C. Antiplane crack problems for an inhomogeneous elastic material. *Acta mech* 1978;**29**(1–4):199–211.
3. Gerasoules A, Srivastav RD. A Griffith crack problem for a nonhomogeneous medium. *Intern J Eng Science* 1980;**18**(1):239–47.
4. Dhaliwal RS, Singh BM. On the theory of elasticity of a nonhomogeneous medium. *Elasticity* 1978;**8**(2):211–9.
5. Schovanec LA. Griffith crack problem for an inhomogeneous elastic material. *Acta mech* 1986;**58**(1–2):67–80.
6. Singh- BM, Dhahwal RS. Griffith crack in an infinite nonhomogeneous elastic medium under shear. *Appl Sci* 1978;**34**(1):17–24.
7. Sudhu Singh, Singh BM. A Griffith crack problem in a infinite nonhomogeneous elastic medium. *J InstEngin* 1979;**59**(1):177–9.
8. Delate E, Erdogan F. The crack problem for a nonhomogeneous plane. *Trans ASME J Appl Mech* 1983;**50**(3):609–14.
9. Sneddon LN. The distribution of stress in the neighbourhood of a crack in an elastic solid. *Proc Roy Soc London Ser A* 1946;**187**(1008):229–60.
10. Sack RA. Extension of Griffith's theory of rupture to three dimensions. *Proc Phys Soc London* 1941;**58**(330):729–36.
11. Erdogan F. Stress distribution in bonded dissimilar materials containing circular or ring shaped cavities. *Trans ASME Ser E J Appl Mech* 1965;**32**(4):829–36.
12. Kassir MK, Bregman AM. The stress intensity factor for a penny-shaped crack between two dissimilar media. *Trans ASME J Appl Mech* 1972;**94**(1):308–10.
13. Lowengrub M, Sneddon LN. The effect of shear on a penny-shaped crack at the interface of an elastic half space and a rigid foundation. *Intern J Engg Sci* 1972;**10**(10):899–913.
14. Willis JR. The penny-shaped crack on an interface. *Quart J Mech Appl* 1972;**2.5**(3):367–85.
15. Erdogan E, Arin K. Penny-shaped interface crack between an elastic layer and a half space. *Intern J Engg Sci* 1972;**10**(2):115–25.
16. Kassir MK, Sih GC. *Three-Dimensional Crack Problems*. Leyden: Noordhoff; 1975, 452 p.
17. Panasyuk VV, Andrejkin AE, Stadnik MM. Tree-dimensional static crack problems solutions (a review). *Eng Fract Mech* 1981;**14**(2):245–60.
18. Gerasoulis A, Srivastav RP. A Griffith crack problem for a non-homogeneous medium. *Intern J Englg Sci* 1980;**18**(1):239–47.
19. Delate E, Erdogan F. The crack problem for a nonhomogeneous plane. *Trans ASME J Appl Mech* 1983;**50**(3):609–14.
20. Pal S, Kulkarni MG, Kubair DV. Mode-3 spontaneous crack propagation in symmetric functionally graded materials. *Intern J Solids and Structures* 2007;**44**(1):242–54.
21. *Functionally Gradient Materials.* Symp./Eds J.B. Holt et al. Westerville, Ohio: Amer. Ceramic Soc, 1993. 469 p. (Ceramic Transactions. Vol. 34).
22. Fett T, Munz D. A weight function for cracks in gradient materials. *Intern J Fracture* 1997;**84**(1):L3–7.
23. Selvadurai APS. The penny-shaped crack at a bonded plane with localized elastic non-homogeneity. *Eur J Mech A/Solids* 2000;**19**(3):525–34.

24. Erdogan F, Gupta G. The stress analysis of multi-layered composites with a flaw. *Intern J Solids and Structures* 1971;**7**(1):39–61.
25. Erdogan F, Gupta G. Layered composites with an interface flaw. *Itnern J Solids and Structures* 1971;**7**(8):1089–107.
26. Sih GC, Chen EP. Crack in composite materials. In: Sih GC, editor. *Mechanics of Fractire*, 6. Hague: Martinus Nijhoff Publi; 1981. p. 15–81.
27. Aizikovich SM, Trubchik IS. Asymptotic solutions of the crack problem for a non-homogeneous elastic material. In: *Fracture Mechanics Successes and Problems. Abstracts. 8th Intern. Conf. Fracture*. 1993. Pt. 1. P. 114.
28. Borodachev AN. Equilibrium of a non-homogeneous elastic body with a circular crack. *Prikl Mekh* 1986;**22**(2):108–13.
29. Borodachev AN. A method of constructing weighting functions for a circular crack. *Prikl Mat Mekh* 1990;**54**(6):1022–30.
30. Aizikovich SM, Aleksandrov VM. The axisymmetric problem of the imbedding of a circular punch in an elastic half-space inhomogeneous in depth. *Izv Akad Nauk SSSR MTT* 1984;**2**:73–82.
31. Aizikovich SM, Aleksandrov VM. The properties of compliance functions corresponding to a layered and continuously inhomogeneous half-space. *Dokl Akad Nauk SSSR* 1982;**266**(1):40–3.
32. Privarnikov AK. Spatial deformation of a multilayer foundation. In: *The Stability and Strength of structural Components.* Dnepropetrovsk: Dnepropetrovsk Univ; 1973, 27–45.
33. Ishlinskii AYU. *Applied Problems of Mechanics*, Vol.2. Moscow: Nauka; 1986.
34. Koshlyakov NS, Gliner EB, Smirnov MM. *Differential Equations of Mathematical Physics*. Moscow: Fizmatgiz; 1962.
35. Aizikovich SM. Asymptotic solutions of contact problems of elasticity theory for media inhomogeneous in depth. *Prikl Mat Mekh* 1982;**46**(1):148–58.
36. Aizikovich SM. Asymptotic solution of one class of dual equations for small values of the parameter. *Dokl Akad Nauk SSSR* 1990;**313**(1):48–52.
37. Aizikovich SM. Asymptotic solution of one class of dual equations for large values of the parameter. *Dokl Akad Nauk SSSR* 1991;**319**(5):1037–41.
38. Kantorovich LV, Akilov GP. *Functional Analysis*. Moscow: Nauka; 1977.
39. Eischen JW. Fracture of nonhomogeneous materials. *Intern J Fract* 1987;**34**(1):3–22.

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